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Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations

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Abstract

Perturbation methods depend on a small parameter which is difficult to be found for real-life nonlinear problems. To overcome this shortcoming, two new but powerful analytical methods are introduced to solve nonlinear heat transfer problems in this article; one is He's variational iteration method (VIM) and the other is the homotopy-perturbation method (HPM). The VIM is to construct correction functionals using general Lagrange multipliers identified optimally via the variational theory, and the initial approximations can be freely chosen with unknown constants. The HPM deforms a difficult problem into a simple problem which can be easily solved. Nonlinear convective–radiative cooling equation, nonlinear heat equation (porous media equation) and nonlinear heat equation with cubic nonlinearity are used as examples to illustrate the simple solution procedures. Comparison of the applied methods with exact solutions reveals that both methods are tremendously effective.

Keywords: Heat transfer; Nonlinear equations; Variational iteration method (VIM); Homotopy-perturbation method (HPM)

1. Introduction

Most scientific problems and phenomena such as heat transfer occur nonlinearly. Except in a limited number of these problems, we have difficulty in finding their exact analytical solutions. Therefore, approximate analytical solutions are searched were introduced, among which variational iteration method (VIM) [7–9,11,19] and homotopy-perturbation method (HPM) [10,12–16] are the most effective and convenient ones for both weakly and strongly nonlinear equations.

Perturbation method [3] provides the most versatile tools available in nonlinear analysis of engineering problems, but its limitations hamper its application [27]:

1. Perturbation method is based on assuming a small parameter. An overwhelming majority of nonlinear problems, especially those having strong nonlinearity, have no small parameters at all.
2. The approximate solutions obtained by the perturbation methods, in most cases, are valid only for the small values of the small parameter. The perturbation solutions are generally uniformly valid as long as a specific system parameter is small. However, we cannot rely fully on the approximations, because there is no criterion on which the small parameter should exist. Thus, it is essential to check the validity of the approximations numerically and/or experimentally.

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To overcome these difficulties, some new methods have been proposed recently, a complete review on various asymptotic methods can be found in [2,27]. This paper will apply the VIM [7–9,11,19] and the HPM [10,12–16] to some nonlinear heat transfer equations. Recently, some rather extraordinary virtues of the both methods have been explained, and there has been a considerable deal of research in applying the methods for solving various strongly nonlinear equations [1,4–6,17,18,20,21,25,26].

2. Variational iteration method

To clarify the basic ideas of He's VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \tag{1}$$

where L is a linear operator, N a nonlinear operator and $g(t)$ an inhomogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi))\,\mathrm{d}\xi, \tag{2}$$

where λ is a general Lagrangian multiplier [7–9,11,19] which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation [7–9,11,19], i.e., $\delta\tilde{u}_n = 0$.

3. The application of VIM and HPM in heat transfer

In order to assess the accuracy of VIM for solving nonlinear equations and to compare it with HPM, we will consider the two following examples.

3.1. Cooling of a lumped system by combined convection and radiation (Example 1)

Consider the problem of combined convective–radiative cooling of a lumped system [3]. Let the system have volume V, surface area A, density ρ, specific heat c, emissivity E and the initial temperature T_i. At $t = 0$, the system is exposed to an environment with convective heat transfer with the coefficient of h and the temperature T_a. The system also loses heat through radiation and the effective sink temperature is T_s. The cooling equation and the initial conditions are as follows:

$$\rho V c \frac{\mathrm{d}T}{\mathrm{d}t} + hA(T - T_a) + E\sigma A(T^4 - T_s^4) = 0, \tag{3a}$$

$$t = 0, \quad T = T_i. \tag{3b}$$

To solve the equation, we do the following changes of parameters:

$$\theta = \frac{T}{T_i}, \quad \theta_a = \frac{T_a}{T_i}, \quad \theta_s = \frac{T_s}{T_i}, \quad \tau = \frac{t}{\rho V c_a/hA}, \quad \varepsilon = \frac{E\sigma T_i^3}{h}. \tag{4}$$

After parameter change, the heat transfer equation will result the following:

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + (\theta - \theta_a) + \varepsilon(\theta^4 - \theta_s^4) = 0, \tag{5a}$$

$$\tau = 0, \quad \theta = 1. \tag{5b}$$

For simplicity, we assume the case $\theta_a = \theta_s = 0$. So we have

$$\frac{\mathrm{d}\theta}{\mathrm{d}\tau} + \theta + \varepsilon\theta^4 = 0, \tag{6a}$$

$$\tau = 0, \quad \theta = 1. \tag{6b}$$

3.1.1. Variational iteration method

In order to solve Eq. (6) using VIM, we construct a correction functional, as follows:

$$\theta_{n+1}(\tau)=\theta_n(\tau)+\int_0^{\tau}\lambda\left\{\frac{\mathrm{d}\theta_n(t)}{\mathrm{d}t}+\theta_n(t)+\varepsilon\tilde{\theta}_n^4(t)\right\}\mathrm{d}t. \tag{7}$$

Its stationary conditions can be obtained as follows:

$$\lambda'(t)-\lambda(t)=0, \tag{8a}$$

$$1+\lambda(t)|_{t=\tau}=0. \tag{8b}$$

The Lagrangian multiplier can therefore be identified as

$$\lambda=-\mathrm{e}^{t-\tau}. \tag{9}$$

As a result, we obtain the following iteration formula:

$$\theta_{n+1}(\tau)=\theta_n(\tau)-\int_0^{\tau}\mathrm{e}^{t-\tau}\left\{\frac{\mathrm{d}\theta_n(t)}{\mathrm{d}t}+\theta_n(t)+\varepsilon\theta_n^4(t)\right\}\mathrm{d}t. \tag{10}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$\theta_0(\tau)=\mathrm{e}^{-\tau}. \tag{11}$$

Using the above variational formula (10), we have

$$\theta_1(\tau)=\theta_0(\tau)-\int_0^{\tau}\mathrm{e}^{t-\tau}\left\{\frac{\mathrm{d}\theta_0(t)}{\mathrm{d}t}+\theta_0(t)+\varepsilon\theta_0^4(t)\right\}\mathrm{d}t. \tag{12}$$

Substituting Eq. (11) into Eq. (12) and after simplifications, we have

$$\theta_1(\tau)=\mathrm{e}^{-\tau}-\tfrac{1}{3}\varepsilon\mathrm{e}^{-\tau}+\tfrac{1}{3}\varepsilon\mathrm{e}^{-4\tau}. \tag{13}$$

In the same way, we obtain $\theta_2(\tau)$ as follows:

$$\begin{aligned}\theta_2(\tau)={}&\mathrm{e}^{-\tau}-\tfrac{1}{3}\varepsilon\mathrm{e}^{-\tau}+\tfrac{1}{3}\varepsilon\mathrm{e}^{-4\tau}+\tfrac{2}{9}\varepsilon^2\mathrm{e}^{-\tau}+\tfrac{1}{81}\varepsilon^4\mathrm{e}^{-\tau}-\tfrac{2}{27}\varepsilon^3\mathrm{e}^{-\tau}-\tfrac{1}{1215}\varepsilon^5\mathrm{e}^{-\tau}\\&-\tfrac{4}{9}\varepsilon^2\mathrm{e}^{-4\tau}+\tfrac{2}{27}\varepsilon^4\mathrm{e}^{-7\tau}-\tfrac{2}{9}\varepsilon^3\mathrm{e}^{-7\tau}-\tfrac{4}{81}\varepsilon^4\mathrm{e}^{-4\tau}+\tfrac{2}{9}\varepsilon^3\mathrm{e}^{-4\tau}+\tfrac{2}{27}\varepsilon^3\mathrm{e}^{-10\tau}+\tfrac{2}{9}\varepsilon^2\mathrm{e}^{-7\tau}\\&+\tfrac{1}{1215}\varepsilon^5\mathrm{e}^{-16\tau}+\tfrac{1}{243}\varepsilon^5\mathrm{e}^{-4\tau}-\tfrac{1}{243}\varepsilon^5\mathrm{e}^{-13\tau}+\tfrac{1}{81}\varepsilon^4\mathrm{e}^{-13\tau}+\tfrac{2}{243}\varepsilon^5\mathrm{e}^{-10\tau}-\tfrac{4}{81}\varepsilon^4\mathrm{e}^{-10\tau}-\tfrac{2}{243}\varepsilon^5\mathrm{e}^{-7\tau}\end{aligned} \tag{14}$$

and so on. In the same manner the rest of the components of the iteration formula can be obtained.

3.1.2. Homotopy-perturbation method

After separating the linear and nonlinear parts of the equation, we apply homotopy-perturbation to Eq. (6), as follows:

$$H(\theta,p)=(1-p)[L(\theta)-L(u_0)]+p[A(\theta)-f(r)]=0, \tag{15}$$

where $L(\theta)$ is the linear part of the equation and $L(u_0)$ the initial approximation [13]. We consider θ as

$$\theta=\theta_0+p\theta_1+p^2\theta_2+\cdots. \tag{16}$$

Substituting Eq. (16) into Eq. (15) and rearranging based on powers of p-terms, we have

$$p^0:\quad \frac{\mathrm{d}\theta_0}{\mathrm{d}\tau}+\theta_0-\frac{\mathrm{d}u_0}{\mathrm{d}\tau}-u_0=0, \tag{17a}$$

$$\tau=0,\quad \theta_0=1, \tag{17b}$$

$$p^1:\quad \frac{\mathrm{d}\theta_1}{\mathrm{d}\tau}+\theta_1+\frac{\mathrm{d}u_0}{\mathrm{d}\tau}+u_0+\varepsilon\theta_0^4=0, \tag{18a}$$

$$\tau = 0, \quad \theta_1 = 0, \tag{18b}$$

$$p^2: \quad \frac{d\theta_2}{d\tau} + \theta_2 + 4\varepsilon\theta_0^3\theta_1 = 0, \tag{19a}$$

$$\tau = 0, \quad \theta_2 = 0. \tag{19b}$$

Solving Eqs. (17)–(19) results in $\theta(\tau)$. When $p \to 1$, we have

$$\theta(\tau) = e^{-\tau} + \tfrac{1}{3}\varepsilon(e^{-4\tau} - e^{-\tau}) - \tfrac{2}{9}\varepsilon^2(-e^{-7\tau} + 2e^{-4\tau} - e^{-\tau}). \tag{20}$$

The exact solution of Eq. (6) is obtained [3] in the following form:

$$\frac{1}{3}\ln\frac{1+\varepsilon\theta^3}{(1+\varepsilon)\theta^3} = \tau. \tag{21}$$

3.2. Porous media equation (Example 2)

As another example, we consider the nonlinear heat equation called the porous media equation [22]:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m \frac{\partial u}{\partial x}\right), \tag{22}$$

where m is a rational number.

This equation often occurs in nonlinear problems of heat and mass transfer, combustion theory and flows in porous media [23]. For instance, it describes unsteady heat transfer in a quiescent medium with the heat diffusivity as a power-law function of a temperature [22].

Let us consider $m = 1$ in Eq. (22). So we have

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) \tag{23}$$

with the initial condition of

$$u(x, 0) = x. \tag{24}$$

3.2.1. Variational iteration method

First we construct a correction functional which reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial}{\partial x}\left(\tilde{u}_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)\right\} d\tau. \tag{25}$$

Its stationary conditions can be obtained as follows:

$$\lambda'(\tau) = 0, \tag{26a}$$

$$1 + \lambda(\tau)|_{\tau=t} = 0. \tag{26b}$$

The Lagrangian multiplier can therefore be identified as

$$\lambda = -1. \tag{27}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial}{\partial x}\left(\tilde{u}_n(x,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)\right\} d\tau. \tag{28}$$

Now we start with an arbitrary initial approximation as follows:

$$u_0(x,t) = x. \tag{29}$$

By the above variational formula (28), we can obtain the following result:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left\{ \frac{\partial u_0(x,\tau)}{\partial \tau} - \frac{\partial}{\partial x}\left(\tilde{u}_0(x,\tau) \frac{\partial u_0(x,\tau)}{\partial x} \right) \right\} \mathrm{d}\tau. \tag{30}$$

Substituting Eq. (29) into Eq. (30) and after some simplifications, we have

$$u_1(x,t) = x + t. \tag{31}$$

In the same way, we obtain $u_2(x,t)$ as

$$u_2(x,t) = x + t. \tag{32}$$

Continuing in this manner, we can obtain that $u_n(x,t) = x + t$ for $n \geqslant 1$, which means that $u(x,t) = x + t$ is the exact solution and can be verified through substitution.

3.2.2. Homotopy-perturbation method

Similar to previous example, after applying HPM and rearranging based on powers of p-terms, we have

$$p^0: \quad \frac{\partial u_0}{\partial t} = 0, \tag{33a}$$

$$u_0(x,0) = x, \tag{33b}$$

$$p^1: \quad \frac{\partial u_1}{\partial t} - \left(\frac{\partial u_0}{\partial x} \right)^2 - u_0 \left(\frac{\partial^2 u_0}{\partial x^2} \right) = 0, \tag{34a}$$

$$u_1(x,0) = 0, \tag{34b}$$

$$p^2: \quad \frac{\partial u_2}{\partial t} - 2\left(\frac{\partial u_0}{\partial x} \right)\left(\frac{\partial u_1}{\partial x} \right) - u_0 \left(\frac{\partial^2 u_1}{\partial x^2} \right) - u_1 \left(\frac{\partial^2 u_0}{\partial x^2} \right) = 0, \tag{35a}$$

$$u_2(x,0) = 0. \tag{35b}$$

Solving Eqs. (33)–(35) results in $u(x,t)$. When $p \to 1$, we have $u(x,t) = x + t$ which is the exact solution of Eq. (23) and can be verified through substitution.

3.3. Nonlinear heat equation with cubic nonlinearity (Example 3)

Finally, we consider the nonlinear heat transfer equation with cubic nonlinearity in the form of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u^3 \tag{36}$$

with the following initial condition:

$$u(x,0) = \frac{1 + 2x}{x^2 + x + 1}. \tag{37}$$

3.3.1. Variational iteration method

First we construct a correction functional which reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \left\{ \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + 2\tilde{u}_n(x,\tau)^3 \right\} \mathrm{d}\tau. \tag{38}$$

Its stationary conditions can be obtained as follows:

$$\lambda'(\tau) = 0, \tag{39a}$$

$$1 + \lambda(\tau)|_{\tau=t} = 0. \tag{39b}$$

The Lagrangian multiplier can therefore be identified as

$$\lambda = -1. \tag{40}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left\{ \frac{\partial u_n(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} + 2\tilde{u}_n(x,\tau)^3 \right\} \mathrm{d}\tau. \tag{41}$$

We start with an initial approximation $u_0(x,t) = u(x,0)$ given by Eq. (37). Using the above variational formula (41), we can obtain the following result:

$$u_1(x,t) = u_0(x,t) - \int_0^t \left\{ \frac{\partial u_0(x,\tau)}{\partial \tau} - \frac{\partial^2 \tilde{u}_0(x,\tau)}{\partial x^2} + 2\tilde{u}_0(x,\tau)^3 \right\} \mathrm{d}\tau. \tag{42}$$

Substituting $u_0(x,t)$ into Eq. (42) and after some simplifications, we have

$$u_1(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2} t. \tag{43}$$

In the same way, we obtain $u_2(x,t)$ as follows:

$$u_1(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2} t + \frac{36(1+2x)}{(x^2+x+1)^3} t^2 \tag{44}$$

and so on. In the same manner the rest of the components of the iteration formula can be obtained and therefore, Eq. (44) can be rewritten in closed form and give the exact solution. This result can be verified through substitution.

3.3.2. Homotopy-perturbation method

After applying HPM and rearranging based on powers of p-terms, we have

$$p^0: \quad \frac{\partial u_0}{\partial t} = 0, \tag{45a}$$

$$u_0(x,0) = \frac{1+2x}{x^2+x+1}, \tag{45b}$$

$$p^1: \quad \frac{\partial u_1}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} + 2u_0^3 = 0, \tag{46a}$$

$$u_1(x,0) = 0, \tag{46b}$$

$$p^2: \quad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} + 6u_0^2 u_1 = 0, \tag{47a}$$

$$u_2(x,0) = 0. \tag{47b}$$

Solving Eqs. (45)–(47) results in $u(x,t)$. When $p \to 1$, we have

$$u(x,t) = \frac{1+2x}{x^2+x+1} - \frac{6(1+2x)}{(x^2+x+1)^2} t + \frac{36(1+2x)}{(x^2+x+1)^3} t^2 \tag{48}$$

which is exactly the same as that obtained by VIM. This result can be verified through substitution.

4. Conclusion

Fig. 1 indicates that the differences among VIM, HPM and the exact solution in example (1) are negligible when the small parameter ε is less than 0.4. In Fig. 2, it is seen that for $\varepsilon = 0.65$, the results of VIM and HPM slightly diverge from the exact solution. This fact has been better shown in Fig. 3; where for $\varepsilon = 0.85$, the differences of the results of

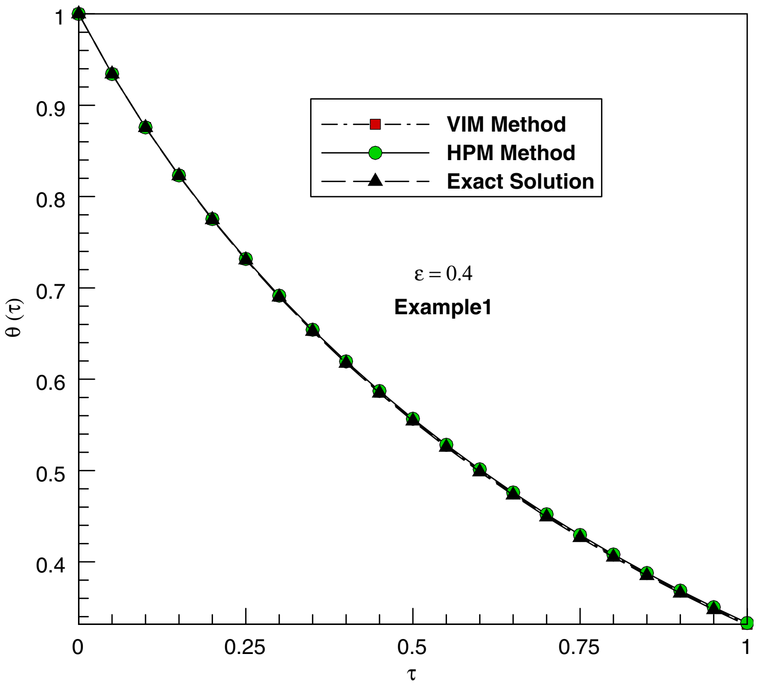

Fig. 1. The comparison of the results of the three methods for Example 1, at $\varepsilon = 0.4$.

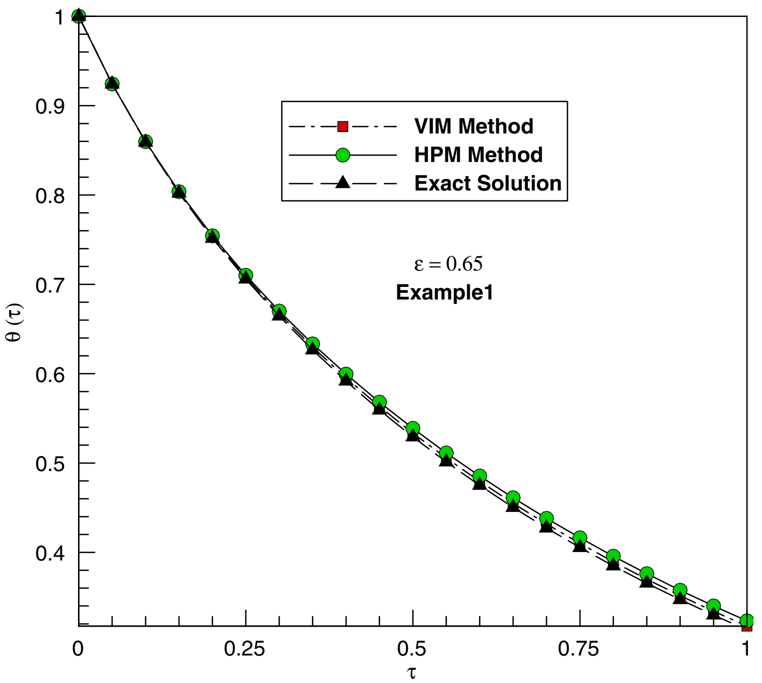

Fig. 2. The comparison of the results of the three methods for Example 1, at $\varepsilon = 0.65$.

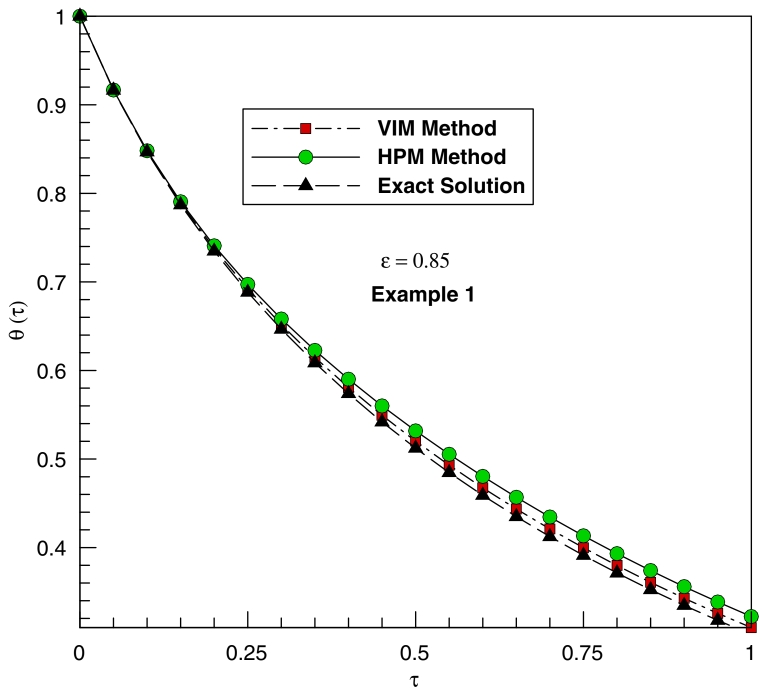

Fig. 3. The comparison of the results of the three methods for Example 1, at $\varepsilon = 0.85$.

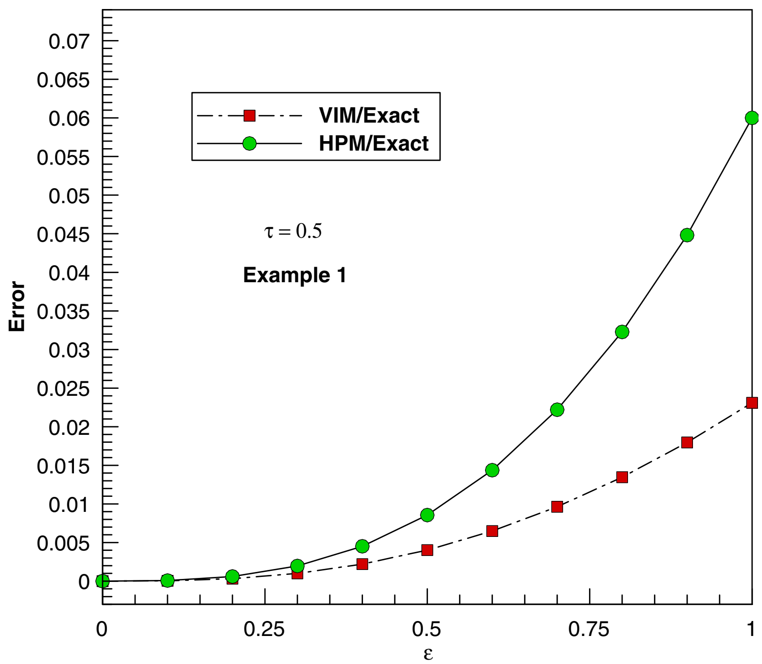

Fig. 4. The comparison of the errors in answers resulted by VIM and HPM for Example 1, at $\varepsilon = 0.5$.

Table 1
The results of VIM and HPM methods and their errors at $\tau = 0.5$

ε	VIM	HPM	Exact	Error of VIM	Error of HPM
0	0.606531	0.606531	0.606531	1.64872E − 10	1.64872E − 10
0.1	0.591617	0.591638	0.591591	4.28955E − 05	7.80444E − 05
0.2	0.578207	0.578371	0.578023	0.000318544	0.000602619
0.3	0.566185	0.566732	0.56562	0.000999783	0.00196688
0.4	0.55544	0.55672	0.554217	0.002207405	0.004516592
0.5	0.545868	0.548335	0.543681	0.004021512	0.008559256
0.6	0.537369	0.541576	0.533903	0.006490236	0.014371001
0.7	0.52985	0.536445	0.524793	0.009636553	0.022201898
0.8	0.523226	0.53294	0.516275	0.013463676	0.032280117
0.9	0.517412	0.531062	0.508284	0.017959388	0.04481523
1	0.512333	0.530812	0.500765	0.023099547	0.060000859

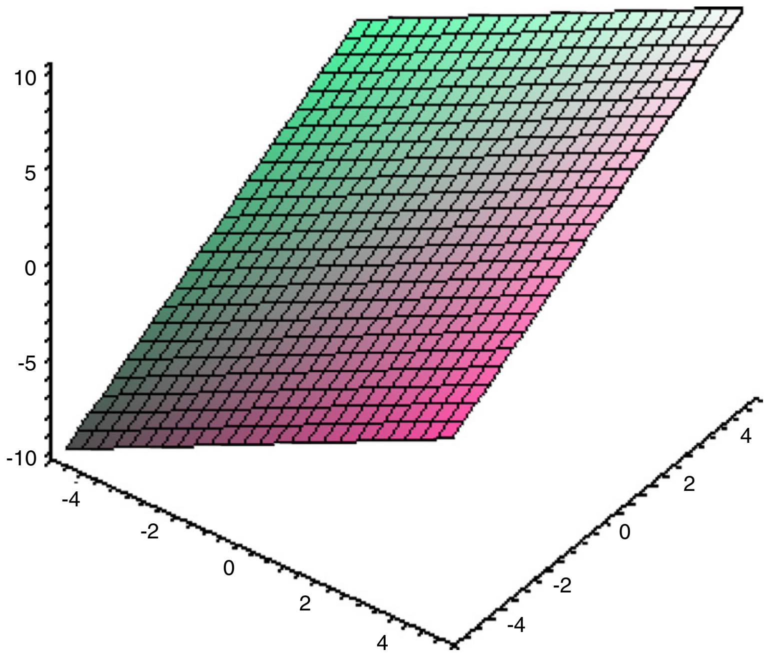

Fig. 5. The exact solution of Eq. (23) resulted by VIM and HPM.

these three methods are more obvious. Considering these three figures, we find out that VIM leads to more acceptable results even for large ε, which is obvious in Fig. 4 where the errors of VIM and HPM are shown and as indicated by Table 1, errors of VIM are less than those of HPM even for large epsilons.

In the second example, VIM and HPM are applied to obtain the exact solution for a nonlinear equation called the porous media equation. It is clear that these methods avoid linearization and are easily applied to obtain the exact solution for this nonlinear equation. Fig. 5 shows the exact solution of Eq. (23) obtained by VIM and HPM.

Finally we applied VIM and HPM to a nonlinear heat equation with cubic nonlinearity. As it is seen, the approximate solution of this equation is easily obtained using VIM and HPM. Fig. 6 shows the solution obtained by the two methods.

In conclusion, VIM and HPM provide highly accurate numerical solutions for nonlinear problems in comparison with other methods. They also do not require large computer memory and discretization of variables t and x. As it is mentioned, this method avoids linearization and physically unrealistic assumptions.

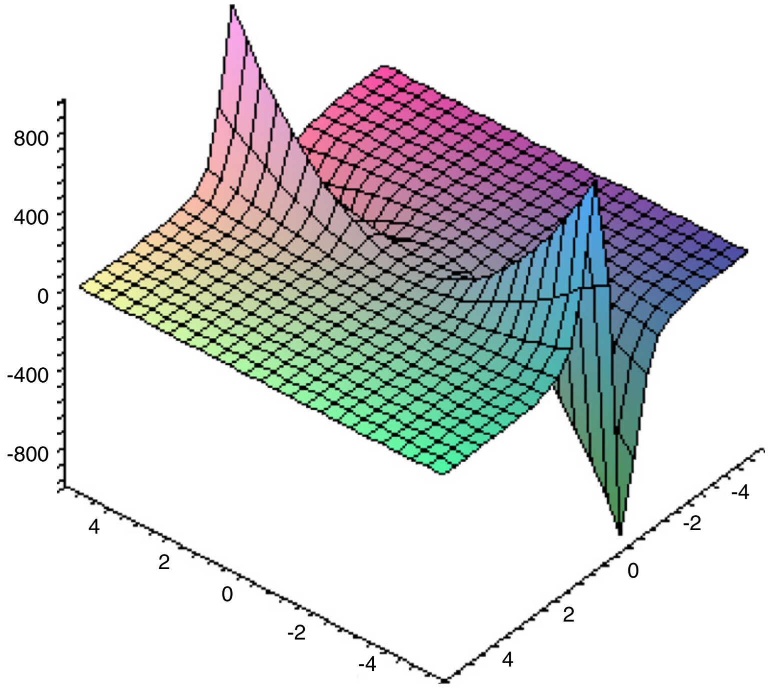

Fig. 6. The solution of Eq. (36) resulted by VIM and HPM.

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Further Reading

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